

The Kodama state for topological quantum field theory beyond instantons

R. Cartas-Fuentevilla and J. F. Tlapanco-Limón

Instituto de Física, Universidad Autónoma de Puebla, Apartado postal J-48 72570 Puebla Pue., México

Constructing a symplectic structure that preserves the ordinary symmetries and the topological invariance for topological Yang-Mills theory, it is shown that the Kodama (Chern-Simons) state traditionally associated with a topological phase of unbroken diffeomorphism invariance for instantons, exists actually for the complete topological sector of the theory. The case of gravity is briefly discussed.

I. Introduction

As is well known Yang-Mills theory in four dimensions formally admits the so called Chern-Simons wavefunction as an exact zero energy eigenfunction of the Schrödinger equation [1]. However, such a state is neither normalizable, nor invariant under CPT. Additionally negative helicity states not only have negative energy but also negative norm, and therefore the Chern-Simons state is not admissible as the ground state of the quantum Yang-Mills theory [2]. A similar situation is found in the context of loop quantum gravity with the so called Kodama state[3], which is the only solution for the quantum Ashtekar constraints [4]. Despite these properties, it is important to understand what these intriguing states describe.

Recently a deep relationship between topological quantum field theory and those states has been found [5]. In the special case of instantons, it has been shown that such states describe topological phases of unbroken diffeomorphism invariance. The self-duality conditions on the fields associated with instantons play a key role producing the deformation of the original actions into topological actions, picking up thus a topological phase among various ground states of those theories. Those states turn out to be the only quantum states for the topological quantum field theories obtained [5, 6]. Along the same lines, a close relationship between self-duality and the Kodama state is found also for Abelian gauge theory in [7]. Therefore, it is natural to associate the Chern-Simons and the Kodama states with the sector corresponding to instantons.

However, in this letter, we shall show that this is not necessarily the case, and we shall find that the Chern-Simons state is associated actually with the whole of the topological sector of the Yang-Mills theory, without invoking the self-duality conditions for instantons, provided that we start from the appropriate topological action.

This work is organized as follows. In the next section we give an outline of the ordinary Yang-Mills

theory and the topological version. In Section III, a symplectic structure for topological Yang-Mills(TYM) theory is constructed. In Section IV the symmetries of this geometrical structure are considered. For completeness in Section V the ordinary quantum Yang-Mills theory and the Chern-Simons state are briefly discussed. Using the symplectic structure previously constructed , in Section VI the classical and quantum TYM theory are analyzed. It is shown also that the corresponding quantum Hamiltonian admits the Chern-Simons wave-function as an eigenfunction with zero energy. The constraints are considered in Section VII, and we finish in Section VIII with some concluding remarks.

II. Ordinary and TYM theory

The usual Yang-Mills action is given by

$$S_{YM}(A) = \alpha \int_M \text{Tr} (F \wedge {}^* F), \quad (1)$$

where α is a parameter, A is the gauge connection, and $F = dA + A \wedge A$ its curvature; d and \wedge correspond to the exterior derivative and the wedge product on M , which we assume as the four-dimensional Minkowski spacetime. *F is the dual of F in the usual sense. Due to this duality operation, the Yang-Mills action depends on the metric structure on M .

The action (1) implies the Yang-Mills equations

$$d{}^*F + [A, {}^*F] = 0, \quad (2)$$

with the curvature F satisfying the Bianchi identity

$$dF + [A, F] = 0. \quad (3)$$

As well known, when the connection is self-dual or anti-self-dual $F = \pm {}^*F$, the Bianchi identity (3) automatically implies the Yang-Mills equations (2), and additionally the action (1) turns out to be a topological action, independent on the metric of M .

Alternatively, one can construct a TYM theory without invoking the self-duality condition,

$$S_{TYM}(A) = \beta \int_M \text{Tr} (F \wedge F), \quad (4)$$

which does not depend on the metric on M , however, it does on the smooth structure of M . Thus,

$$\frac{\delta S_{TYM}}{\delta g_{\mu\nu}} = 0, \quad (5)$$

where $g_{\mu\nu}$ are the components of the metric on M . The action (4) implies trivially the equations (3), *i.e.* every gauge connection A is a critical point for this action. The action (4) is the subject of the present study.

III. The symplectic structure for TYM theory

Following Appendix A in [8], one can construct from the action (4) a symplectic structure that preserves all relevant symmetries of the theory. The variation of the action (4) is given by

$$\delta S_{TYM}(A) = \int_M \text{Tr} \partial_\mu [4\beta \tilde{F}^{\mu\nu} \delta A_\nu] d^4x - 4\beta \int_M \text{Tr} (D_\mu \tilde{F}^{\mu\nu}) \delta A_\nu d^4x, \quad (6)$$

where we have displayed explicitly the components in order to make direct contact with [2, 5]. From (6) we identify the Bianchi identity (3) in terms of components,

$$D_\mu \tilde{F}^{\mu\nu} \equiv \partial_\mu \tilde{F}^{\mu\nu} + [A_\mu, \tilde{F}^{\mu\nu}] = 0, \quad (7)$$

and the argument of the total divergence as a *symplectic potential* for the theory

$$\Theta^\mu \equiv 4\beta \text{Tr} \tilde{F}^{\mu\nu} \delta A_\nu, \quad (8)$$

whose variations give the integral kernel of the symplectic structure:

$$\omega = \int_\Sigma 4\beta \text{Tr} \delta(\tilde{F}^{\mu\nu} \delta A_\nu) d\Sigma_\mu = \int_\Sigma 4\beta \text{Tr} \delta \tilde{F}^{\mu\nu} \delta A_\nu d\Sigma_\mu, \quad (9)$$

where Σ is a Cauchy hypersurface, and we have considered that δ corresponds to the exterior derivative on the phase space Z of the theory [9], which is understood as a submanifold of the *kinematic space* (the space of smooth connections A and curvatures F), on which the *constrictions* (7) hold. It is important to mention that Z has not *a priori* a symplectic structure, for example (9), but an action principle is a necessary ingredient for that [10].

Furthermore, using the first-order variation of Eq. (7) given by

$$D_\mu \delta \tilde{F}^{\alpha\mu} + [\delta A_\mu, \tilde{F}^{\alpha\mu}] = 0,$$

we find that the integral kernel of ω is covariantly conserved, $\partial_\mu (\text{Tr} \delta \tilde{F}^{\mu\nu} \delta A_\nu) = 0$, which makes ω independent on the choice of Σ .

IV. Symmetries of ω

Let us show that ω retains the symmetries of the topological action (4). Since $\delta \tilde{F}^{\mu\nu}$ and δA_ν transform homogeneously under the infinitesimal gauge transformation

$$A_\mu \rightarrow A_\mu + D_\mu \varepsilon, \quad (10)$$

ω is gauge invariant. Furthermore, let \hat{Z} be the phase space Z modulo the action of the symmetry group, and let us show that ω has not components tangent to the gauge directions, which are specified by

$$\delta A'_\mu \rightarrow \delta A_\mu + D_\mu \varepsilon; \quad (11)$$

ω will undergo the transformation

$$\omega' = \omega + \int_\Sigma \partial_\mu \text{Tr} [\tilde{F}^{\mu\nu} \varepsilon] d\Sigma_\nu, \quad (12)$$

where Eqs. (7) have been considered; hence, Eq. (12) shows, for fields with compact support, that ω is a gauge invariant symplectic structure on \widehat{Z} .

On the other hand, the action (4) possesses the topological invariance in the sense of (5). It is straightforward to show that ω is a topological invariant in the same sense, since its expression (9) in terms of components shows that ω does not depend on the metric of M :

$$\omega = \int_{\Sigma} 4\beta \operatorname{Tr} \epsilon^{\mu\nu\alpha\beta} \delta F_{\alpha\beta} \delta A_{\nu} d\Sigma_{\mu}, \quad (13)$$

which can be expressed in a compact form as

$$\omega = \int_{\Sigma} 4\beta \operatorname{Tr} \delta F \wedge \delta A, \quad (14)$$

which shows clearly the independence of ω on the metric structure of M , such as the action itself in (4).

Thus

$$\frac{\delta\omega}{\delta g_{\mu\nu}} = 0. \quad (15)$$

However, as well known, the topological action (4) is not invariant under the parity operation P , and thus CP and CPT are not symmetries for such an action. From the expressions (9) or (14), it is easy to note that ω inherits this property.

V. The Chern-Simons state for ordinary YM theory

For completeness we give an outline of what is known about the Chern-Simons state in conventional YM theory.

The quantum Hamiltonian for Yang-Mills theory can be obtained from the classical expression for the energy,

$$H = \frac{1}{2g^2} \int d^3x \operatorname{Tr} (E^2 + B^2) = \frac{1}{2} \int d^3x \operatorname{Tr} \left(-g^2 \frac{\delta^2}{\delta A(x)^2} + \frac{1}{g^2} B^2 \right), \quad (16)$$

where g is the gauge coupling, $E_i = F_{0i}$, $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$, and quantum mechanically the canonical momentum $\frac{E}{g^2}$ becomes $-i\frac{\delta}{\delta A}$:

$$\frac{F_{0i}}{g^2} \rightarrow -i\frac{\delta}{\delta A_i}. \quad (17)$$

The Chern-Simons functional

$$I = \frac{1}{4\pi} \int d^3x \operatorname{Tr} (\epsilon^{ijk} (A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k)), \quad (18)$$

for which $\frac{\delta I}{\delta A} = \frac{B}{2\pi}$, allows to construct the wave-function

$$\psi(A) = \exp\left[\left(\frac{2\pi}{g^2}\right)I(A)\right], \quad (19)$$

satisfying

$$(E + iB)\psi = i\left(-g^2 \frac{\delta}{\delta A} + B\right)\psi = 0, \quad (20)$$

and thus ψ corresponds to an eigenstate of the Hamiltonian (16) with zero energy.

VI. Classical and Quantum Hamiltonian for TYM theory and the Chern-Simons state

We can obtain the physical content of the symplectic structure ω following [9], and to consider the contraction of ω with the phase space vector field V corresponding to the translation by a constant spacetime vector ε^μ . Considering that

$$V \rfloor \delta A^\mu = \varepsilon_\beta F^{\beta\mu},$$

we have

$$V \rfloor \delta \tilde{F}^{\alpha\mu} = \varepsilon^\beta D_\beta \tilde{F}^{\alpha\mu},$$

and hence, using Eq. (9),

$$V \rfloor \omega = \varepsilon_\beta \int d\Sigma_\alpha \text{Tr} \delta(-4\beta) [F^{(\beta}{}_\mu \tilde{F}^{\alpha)\mu} - \frac{1}{4} n^{\alpha\beta} F_{\lambda\mu} \tilde{F}^{\lambda\mu}],$$

modulo total divergences. Therefore, from the above equation we can identify the (symmetric and gauge-invariant) energy-momentum tensor,

$$T^{\mu\nu} = -\text{Tr} \left(4\beta (F^{(\mu}{}_\alpha \tilde{F}^{\nu)\alpha} - \frac{1}{4} g^{\mu\nu} \tilde{F}^{\mu\nu} F_{\mu\nu}) \right), \quad (21)$$

which is classically zero, as expected for a topological action, in concordance with Eq. (5). From Eq. (21) we can identify the energy density for the theory,

$$T^{00} = -2\text{Tr} \left(\beta (F_{0i} \tilde{F}^{0i} - \frac{1}{2} F_{ij} \tilde{F}^{ij}) \right) = \text{Tr} F_{0i} (2\beta \tilde{F}^{0i} - \beta \epsilon_{ijk} F_{jk}) = \text{Tr} F_{0i} (\pi_i - \beta \epsilon_{ijk} F_{jk}),$$

where $\tilde{F}^{ij} = \epsilon^{ijk} F_{0k}$, and $\pi_i \equiv 2\beta \tilde{F}^{0i}$; hence the classical Hamiltonian is given by

$$H = \int_\Sigma d\Sigma \text{Tr} F_{0i} (\pi_i - \beta \epsilon_{ijk} F_{jk}). \quad (22)$$

The idea is to use the symplectic structure ω constructed previously in Section III for obtaining from (22) the corresponding quantum Hamiltonian. Therefore, considering that $d\Sigma_\mu$ is a time-like vector field, we can obtain in particular the following (non-covariant) description of the phase space,

$$\omega = \int_\Sigma 4\beta \text{Tr} (\delta \tilde{F}^{0i} \wedge \delta A_i), \quad (23)$$

in order to make contact with [2, 5]. This expression for the symplectic structure will allow us to work in the *temporal gauge* $A_0 = 0$. Furthermore, Eq. (23) shows explicitly that the canonical variables for TYM theory are given by $(2\beta) \tilde{F}^{0i}$ and A_i . Therefore, we have the classical-quantum correspondence,

$$(2\beta) \tilde{F}^{0i} \rightarrow i \frac{\delta}{\delta A_i}. \quad (24)$$

Making a comparison with Eq. (17), we see that the canonical momentum for TYM theory is the dual of that for conventional YM theory. In the case of self-dual fields, the correspondence (24) reduces to that in Eq. (17), and similarly the expressions (23), (22), and (21) reduce to the corresponding expressions given in Section V, in the same sense that the topological action (4) becomes that in Eq. (1). Thus we have, as a particular scenario, the self-dual case. However, we avoid at all, as already mentioned in the introduction, the *temptation* of imposing such a condition, and we shall prove that the Chern-Simons state is an eigenstate with zero energy for TYM theory without invoking the self-dual condition.

In this manner, considering Eq. (24), the quantum Hamiltonian will be given by

$$H_Q = \int_{\Sigma} d\Sigma \operatorname{Tr} F_{0i} \left(i \frac{\delta}{\delta A_i} - \beta \epsilon_{ijk} F_{jk} \right). \quad (25)$$

In the temporal gauge $A_0 = 0$, it is easy to show that $[F_{0i}, \frac{\delta}{\delta A_i}] = 0$, and thus we have no ordering ambiguity in the quantum Hamiltonian (25). Thus, any wave function ψ satisfying

$$\left(i \frac{\delta}{\delta A_i} - \beta \epsilon_{ijk} F_{jk} \right) \psi = 0, \quad (26)$$

will correspond to a state of zero energy for the Hamiltonian (25). Therefore, the solution for Eq. (26) is given by

$$\psi(A) = e^{-4\pi i \beta I(A)}, \quad (27)$$

with $I(A)$ given in Eq. (18). Equation (27) is essentially the Chern-Simons wave function (19), and we conclude then that the Chern-Simons state corresponds strictly to a (topological) state of the TYM theory, without invoking the self-duality condition on the fields. Thus, such a state is associated with the topological phase of unbroken diffeomorphism invariance of the complete topological sector.

VII. The constraints

Equations (7) imply that

$$D_i \tilde{F}^{i0} = 0, \quad (28)$$

which must be considered as a constraint on the quantum states, in particular on the Chern-Simons state. Considering the correspondence (24) for TYM theory, such a constraint reads

$$D_i \frac{\delta}{\delta A_i} \psi = 0, \quad (29)$$

which ensures that ψ is unchanged under *small* gauge transformations [1]. Note that for self-dual fields, Eq. (28) reduces to the Gauss law for conventional YM theory, and (29) will represent the usual Gauss law constraint. However, as we have shown, the constraint (29) and its relationship with invariance of ψ under small gauge transformations exist in the whole of the topological sector of the theory and, in particular, for instantons. In a more familiar way, Eq. (28) corresponds classically to the equation $D_i B^i = 0$, and thus we can identify it as the generator of gauge symmetries for TYM theory at a quantum level in accordance with Eq. (29).

On the other hand, within the Dirac quantization scheme, the correspondence (24) gives rise to the *primary* constraint

$$i \frac{\delta}{\delta A_i} - 2\beta \tilde{F}_{0i} = i \frac{\delta}{\delta A_i} - \beta \epsilon_{ijk} F_{jk} \approx 0, \quad (30)$$

which is of first-class, and thus any physical quantum state ψ must satisfy

$$(i \frac{\delta}{\delta A_i} - \beta \epsilon_{ijk} F_{jk}) \psi = 0, \quad (31)$$

that corresponds exactly to Eq. (26) for the Chern-Simons state. In this sense, the Hamiltonian (25) is purely a combination of the constraints (31), with F_{0i} , the dual of the canonical momentum, playing the role of a Lagrange multiplier field.

VIII. Concluding remarks

It is important to remark at this point the results obtained. The starting point is the topological Yang-Mills theory, which has as *equations of motion* the Bianchi identities, and as *solution space* the complete space of gauge connections. On this solution space the topological action defines a symplectic structure, and the corresponding Hamiltonian admits the Chern-Simons wavefunction as a zero energy eigenfunction. Hence, the Chern-Simons state exists for the complete topological sector of the theory, and in order to establish its existence, neither the self-dual condition nor the Yang-Mills equations are required. The Bianchi identity becomes the generator of gauge symmetries at quantum level. As a particular case, the topological phases of Yang-Mills theory can be obtained invoking self-duality.

The subject of references [2, 5] is focused on the question *why the Chern-Simons state exists*. The present results suggest that this question should be translated beyond the instantons sector, *i.e.* to the complete topological sector of the theory. Furthermore, the problem of the normalizability of this state under an appropriate inner product must be also reformulated in the scheme of the TYM theory, since the inner product used in the ordinary field theory seems to fail in the topological scheme.

Although instantons play a vital role in fundamental aspects of field theories, it is of vital importance also to clarify those aspects that are not necessarily related with such topological objects. In this sense, the problem of studying the sectors beyond the instantons even persists; it is possible that those aspects normally associated with instantons, are actually related with the whole of the topological sectors.

Following the arguments given in the present treatment for TYM theory, it is possible that in the context of loop quantum gravity similar results can be obtained; however, this will be a problem for future works.

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